PARTIAL SUMS OF MEROMORPHIC FUNCTIONS

Dr. M.APARNA, Sr. Asst.Prof of Mathematics

G.Narayanamma Institute of Technology & Science, Hyderabd.

Abstract: In this paper we discussed the Partial Sums of Meromorphic Functions defined in terms of convex and starlike functions, Silverman determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums.
In this paper we analyze the work of Silverman investigating the ratio of a function

Key Words: Univalent Function, Starlike Function, convex Function, Analytic Function,Meromorphic Functions.

1. INTRODUCTION:

Let $\sum$ be the class consists of functions of the form

(1.1) $$f(z) = \frac{1}{z(p-q)} + \sum_{k=1}^{\infty} a_k z^k$$

which are analytic in the open unit disk $D = \{z : 0 < |z| < 1\}$. Let $\sum^*[\alpha(p,q)]$ and $\sum_k[\alpha(p,q)]$ be the subclasses of $\sum$ consisting of all functions which are meromorphic starlike and meromorphic convex of order $\alpha(p,q)$ $(0 \le \alpha(p,q) < 1)$ in D. We denoted by $\sum_c[\alpha(p,q)]$ the subclass of $\sum$ which satisfies

$$-\operatorname{Re}\{z^2 f'(z)\} > \alpha(p,q) \qquad (0 < \alpha(p,q) < 1; z \in u = D \cup \{0\})$$

We observe that every function belong to the class $\sum_c[\alpha(p,q)]$ is meromorphic close-to-convex of order $\alpha(p,q)$ in D. If $f(z) = \sum_{k=o}^{\infty} a_k z^k$ and $g(z) = \sum_{k=o}^{\infty} b_k z^k$ are analytic in u, then their Hadamard product (or convolution) denoted by $f * g$, is the function defined by the power series.

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \qquad (z \in u)$$

A sufficient condition for a function f of the form (1.1) to be in $\sum^*[\alpha(p,q)]$ is

(1.2) $$\sum_{k=1}^{\infty} [k + \alpha(p,q)] |a_k| \le 1 - \alpha(p,q)$$

and $\sum_k[\alpha(p,q)]$ is

(1.3) $$\sum_{k=1}^{\infty} k[k+\alpha(p,q)]|a_k| < 1-\alpha(p,q)$$

These sufficient conditions are also necessary for functions of the form (1.1) with positive or negative coefficients. Recently Silverman determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Also Li and Owa obtained the radius for which the normalized univalent functions in u, the partial sums of the well known Libera integral operator imply starlikeness. Further for various other interesting developments concerning partial sums of analytic univalent functions referred to the works of Brickman, Sheil-Small, Silvia, Singh and Singh and Yang and Owa.

Since to a certain extent the work in the meromorphic univalent case has paralleled that of the analytic univalent case. One is to search results analogous to those of Silverman for meromorphic univalent functions in D. In this paper we analyze the work of Silverman investigating the ratio of a function of the form (1.1) to its sequence of partial sums $f_n(z) = \dfrac{1}{z(p,q)} + \sum_{k=1}^{n} a_k z^k$ when the coefficients are sufficiently small to satisfy either condition (1.2) or (1.3) We also determined lower bounds for $\operatorname{Re}\{f(z)/f_n(z)(p-q)\}$, $\operatorname{Re}\{f_n(z)/f(z)(p-q)\}$, $\operatorname{Re}\{f'(z)/f_n'(z)(p-q)\}$ and $\operatorname{Re}\{f_n'(z)/f'(z)(p-q)\}$ further ,We gave a property for the partial sums of certain integral operators in connection with meromorphic close-to-convex functions.

We use the well known result that $\operatorname{Re}\{(1+w(z))/(1-w(z))\} > 0 \quad (z \in u)$ if and only if $w(z) = \sum_{k=1}^{\infty} c_k z^k$ satisfies the inequality $|w(z)| < |z|$. We will assume that f is of the form (1.1) and its sequence of partial sums is

$$f_n(z) = \frac{1}{z(p,q)} + \sum_{k=1}^{n} a_k z^k$$

2. Theorem: If f of is the form (1.1) satisfies condition (1.2), then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)(p-q)}\right\} \geq \frac{n+2\alpha(p,q)}{n+1+\alpha(p,q)} \qquad (z \in u).$$

The result is for every n, with extremal function

(2.1) $$f(z) = \frac{1}{z(p,q)} + \frac{1-\alpha(p,q)}{n+1+\alpha(p,q)} z^{n+1} \qquad (n \geq 0)$$

Proof: Let us write

$$\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\left[\frac{f(z)}{f_n(z)}-\frac{n+2\alpha(p,q)}{n+1+\alpha(p,q)}\right]=\frac{1+\sum_{k=1}^{n}a_kz^{k+1}+\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_kz^{k+1}}{1+\sum_{k=1}^{n}a_kz^{k+1}}$$

$$=\frac{1+A(z)}{1+B(z)}$$

We know that

$$\frac{1+A(z)}{1+B(z)}=\frac{1+w(z)}{1-w(z)}$$

so that $$w(z)=\frac{A(z)-B(z)}{2+A(z)+B(z)}$$

Then $$w(z)=\frac{\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_kz^{k+1}}{2+2\sum_{k=1}^{n}a_kz^{k+1}+\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_kz^{k+1}}$$

And $$|w(z)|<\frac{\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k|-\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|}$$

Now $|w(z)<1|$ if and only if

$$2\left(\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\right)\sum_{k=n+1}^{\infty}|a_k|\le 2-2\sum_{k=1}^{n}|a_k|$$

which is equivalent to

(2.2) $$\sum_{k=1}^{n}|a_k|+\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|\le 1$$

It sufficient to show that the left hand side of (2.2) is bounded above by $\sum_{k=1}^{\infty}\left[\frac{k+\alpha(p,q)}{1-\alpha(p,q)}\right]|a_k|$

Which is equivalent to $\sum_{k=1}^{n}\left(\frac{k+2\alpha(p,q)-1}{1-\alpha(p,q)}\right)|a_k|+\sum_{k=n+1}^{\infty}\left(\frac{k-n-1}{1-\alpha(p,q)}\right)|a_k|\ge 0$

The function f is given by (2.1) which gives the result, we observe that for $z = re^{\Pi i/(n+2)}$

$$\frac{f(z)}{f_n(z)(p-q)} = 1 + \frac{1-\alpha(p,q)}{n+1+\alpha(p,q)} z^{n+2} \to 1 - \frac{1-\alpha(p,q)}{n+1+\alpha(p,q)} = \frac{n+2\alpha(p,q)}{n+1+\alpha(p,q)}$$

when $r \to 1^-$

which complete the proof of the theorem.

By putting p=1, q=0, we have the following corollary which

states as follows

3. Corollary: If f is of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

satisfies the condition

$$\sum_{k=1}^{\infty} [k+\alpha] |a_k| \le 1-\alpha$$

then $$\mathrm{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \ge \frac{n+2\alpha}{n+1+\alpha} \qquad (z \in u).$$

The result is for every n with extremal function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{n+1+\alpha} z^{n+1} \qquad (n \ge 0).$$

4. Theorem: If f is of the form (1.1) satisfies the condition (1.3) then

$$\mathrm{Re}\left\{\frac{f(z)}{f_n(z)(p-q)}\right\} \ge \frac{(n+2)(n+\alpha(p,q))}{(n+1)(n+1+\alpha(p,q))} \qquad (z \in u)$$

The result is for every n with extremal function

$$(4.1) \qquad f(z) = \frac{1}{z(p-q)} + \frac{1-\alpha(p,q)}{(n+1)(n+1+\alpha(p,q)} z^{n+1} \qquad (n > 0)$$

Proof: Write

$$\frac{(n+1)(n+1+\alpha(p,q))}{1-\alpha(p,q)} \left[\frac{f(z)}{f_n(z)(p-q)} - \frac{(n+2)(n+\alpha(p,q))}{(n+1)(n+1+\alpha(p,q))}\right]$$

$$=\frac{1+\sum_{k=1}^{n}a_k z^{k+1}+\frac{(n+1)(n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_k z^{K+1}}{1+\sum_{k=1}^{n}a_k z^{k+1}}$$

$$=\frac{1+w(z)}{1-w(z)}$$

Where

$$w(z)=\frac{\frac{(n+1)(n+1+\alpha(p,q))}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_k z^{k+1}}{2+2+\sum_{k=1}^{n}a_k z^{k+1}+\frac{(n+1)(n+1+\alpha(p,q))}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_k z^{k+1}}$$

Now

$$|w(z)| \leq \frac{\frac{(n+1)(n+1+\alpha(p,q))}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k|-\frac{(n+1)(n+1+\alpha(p,q))}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|} \leq 1$$

If

(4.2) $$\sum_{k=1}^{n}|a_k|+\frac{(n+1)(n+1+\alpha(p,q))}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|\leq 1$$

The left hand side of (4.2) is bounded above by $\sum_{k=1}^{\infty}\left[\frac{k(k+\alpha(p,q)}{1-\alpha(p,q)}\right]|a_k|$

If $$\frac{1}{1-\alpha(p,q)}\left\{\sum_{k=1}^{n}\left[k(k+\alpha(p,q)-(1-\alpha(p,q))\right]|a_k|\right.$$

$$\left.+\sum_{k=n+1}^{\infty}\left[k(k+\alpha(p,q))-(n+1)(n+1+\alpha(p,q))\right]|a_k|\right\}\geq 0$$

The proof is completed.

By putting p=1, q=0, we have the following corollary which states as follows

5. Corollary: If f is of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k$$

satisfies the condition

$$\sum_{k=1}^{\infty}k[k+\alpha]|a_k|\le 1-\alpha$$

then $$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}\ge\frac{(n+2)(n+\alpha)}{(n+1)(n+1+\alpha)}\qquad (z\in u).$$

The result is sharp for every n with extremal function

$$f(z)=\frac{1}{z}+\frac{1-\alpha}{(n+1)(n+1+\alpha)}z^{n+1}\qquad (n\ge 0).$$

6. Theorem:

a) If f is of the form (1.1) satisfies the condition (1.2) then

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)(p-q)}\right\}\ge\frac{n+1+\alpha(p,q)}{n+2}\qquad (z\in u)$$

b) If 'f' is of the form (1.1) satisfies the condition (1.3) then

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)(p-q)}\right\}\ge\frac{(n+1)(n+1+\alpha(p,q))}{(n+1)(n+2)-n(1-\alpha(p,q))}\qquad (z\in u)$$

Equalities hold in (a) and (b) for the functions given by (2.1) and (4.1) respectively.

Proof: a) Write

$$\frac{(n+2)}{1-\alpha(p,q)}\left[\frac{f_n(z)}{f(z)(p-q)}-\frac{n+1+\alpha(p,q)}{n+2}\right]$$

$$=\frac{1+\sum_{k=1}^{n}a_k z^{k+1}+\frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}a_k z^{K+1}}{1+\sum_{k=1}^{n}a_k z^{k+1}}$$

$$=\frac{1+w(z)}{1-w(z)}$$

where

$$|w(z)| \leq \frac{\frac{n+2}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k| - \frac{n+2[\alpha(p,q)]}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k|} \leq 1$$

This inequality is equivalent to

$$(6.1) \qquad \sum_{k=1}^{n}|a_k| + \frac{n+1+\alpha(p,q)}{1-\alpha(p,q)}\sum_{k=n+1}^{\infty}|a_k| \leq 1$$

The left hand side of (6.1) is bounded above by

$$\sum_{k=1}^{\infty}\left[\frac{k+\alpha(p,q)}{1-\alpha(p,q)}\right]|a_k|$$

The proof is completed.

The proof of (b) is similar to (a).

By putting p=1, q=0, in the above theorem we have the following corollary which states as follows

7. Corollary:

a) If f is of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

satisfies the condition

$$\sum_{k=1}^{\infty}[k+\alpha]|a_k| \leq 1-\alpha$$

then $$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{n+1+\alpha}{n+2} \qquad (z \in u).$$

b) If f is of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

satisfies the condition

$$\sum_{k=1}^{\infty} k[k+\alpha]|a_k| \leq 1-\alpha$$

then $$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(n+1)(n+1+\alpha)}{(n+1)(n+2)-n(1-\alpha)} \qquad (z \in u).$$

8. Theorem: If f is of the form (1.1) satisfies the condition (1.2) with $\alpha(p,q)=0$ then

a) $$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)(p-q)}\right\} \geq 0 \qquad (z \in u)$$

b) $$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)(p-q)}\right\} \geq \frac{1}{2} \qquad (z \in u)$$

In (a) and (b) the extermal function is given by (2.1) with $\alpha(p,q)=0$.

Proof: The proof is obtained by using the Theorem 2 and (a) of the theorem 6.

By putting p=1, q=0, in the above theorem we have the following corollary which states as follows

9. Corollary: If f is of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

satisfies the condition

$$\sum_{k=1}^{\infty} [k+\alpha]|a_k| \leq 1-\alpha$$

with $\alpha = 0$ then,

a) $$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq 0 \qquad (z \in u).$$

b) $$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq 1/2 \qquad (z \in u).$$

10. Theorm: If f is of the form (1.1) satisfies the condition (1.3) then

a) $$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)(p-q)}\right\} \geq \frac{n+2\alpha(p,q)}{n+1+\alpha(p,q)} \qquad (z \in u)$$

b) $$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)(p-q)}\right\} \geq \frac{n+1+\alpha(p,q)}{n+2} \qquad (z \in u)$$

In (a) and (b) the extremal function is given by (4.1).

Proof: We know that $f \in \sum_k [\alpha(p,q)] \Leftrightarrow -zf' \in \Sigma^*[\alpha(p,q)]$. If 'f' satisfies the condition (1.3) if and only

if -z f' satisfies condition (1.2).Thus (a) is an immediate consequence of Theorem 6.2 and (b) follows directly form (a) of Theorm 4.

For a function $f \in \Sigma$, we define the integral operator 'F' as

$$F(z) = \frac{1}{z^2}\int_0^z tf(t)dt = \frac{1}{z(p-q)} + \sum_{k=1}^{\infty} \frac{1}{k+2} a_k z^k \qquad (z \in D)$$

The n^{th} partial sum F_n of the integral operator F is

$$F_n(z) = \frac{1}{z(p-q)} + \sum_{k=1}^{\infty} \frac{1}{k+2} a_k z^k \qquad (z \in D).$$

Which completes the proof of the theorem.

By putting p=1, q=0, in the above theorem we have the following corollary which states as follows

11. Corollary: If f is of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

satisfies the condition

$$\sum_{k=1}^{\infty} k[k+\alpha]|a_k| \le 1-\alpha$$

then

a) $$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{n+2\alpha}{n+1+\alpha} \qquad (z \in u).$$

b) $$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{n+1+\alpha}{n+2} \qquad (z \in u).$$

12. Lemma: For $0 < \theta < \pi$,

$$\frac{1}{2} + \sum_{k=1}^{m} \frac{\cos(k\theta)}{k+1} \ge 0.$$

This Lemma is due to Rogosinski and Szego .

13. Lemma: Let P be analytic in u with P(0)-1 and Re{P(z)}>1/2 in u. For any function Q analytic in u, the function P*Q takes values in the convex of the image on u under Q. This Lemma is well-known result that can be derived from the Herglotz representation for P.

The above Lemmas 12 and 13 are useful to prove the following Theorem 14.

14. Theorem: If $f \in \sum_c[\alpha(p,q)]$, then $F_n \in \sum_c[\alpha(p,q)]$

Proof: Let f be of the form (1.1) and belong to the class $\sum_c[\alpha(p,q)]$ for $0 \le \alpha(p,q) < 1$. Since $-\operatorname{Re}\{z^2 f'(z)\} > \alpha(p,q)$,

we have

(14.1) $$\operatorname{Re}\left\{1 - \frac{1}{2(1-\alpha(p,q))}\sum_{k=1}^{\infty} k a_k z^{k+1}\right\} > \frac{1}{2} \qquad (z \in u).$$

By using the convolution properties of power series to F_n', we write it as

(14.2) $$-z^2 F_n'(z) - 1 - \sum_{k=1}^{n} \frac{k}{k=2} a_k z^{k+1}$$

Putting $z = re^{i\theta}(0 \le r < 1, 0 \le |\theta| \le \pi)$, and using the minimum principle for harmonic functions along with Lemma 12, we get

(14.3). $$\operatorname{Re}\left\{1 + 2(1-\alpha(p,q))\sum_{k=1}^{n+1} \frac{1}{k+1} z^k\right\} = 1 + 2(1-\alpha(p,q))\sum_{k=1}^{n+1} \frac{r^k \cos k\theta}{k+1}$$

$$> 1 + 2(1-\alpha(p,q))\sum_{k=1}^{n+1} \frac{\cos k\theta}{k+1} \ge \alpha(p,q).$$

By using (14.1), (14.2), (14.3) and Lemma 13,

we proved that

$$\operatorname{Re}\{z^2 F_n'(z)\} > \alpha(p,q) \qquad (0 \le \alpha(p,q) < 1; z \in u).$$

which completes the proof of the theorem.

REFERENCES:

1. Duren P.L.: Univalent Functions, Springer-Verlag Berlin Heidelberg and Tokyo,1983.
2. J.S.Jack: "Functions starlike and convex of order α". J.London Math Soc. 3,(1971), 469 –474.
3. Conway J.B, Functions of complex variables, Narosa Publishing House, 1985.
4. D.R.Wilken and J.Feng, "A remark on convex and starlike functions", Proc.Amer.Math. Soc, (1979), 287-290.
5. O.P.Juneja and T.R.Reddy, "Meromorphic starlike and univalent functions with positive coefficients, Ann. Univ. Mariae Curie-Sklodowsku, 39(1985), 65-76.