A General Model For Fuzzy Linear Programming Problems With Fuzzy Decision Variables And Fuzzy Constraints In The Requirement Vector

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Abstract: In the classical problems of mathematical programming the coefficients and the requirement vector of the problems are assumed to be exactly known. This assumption is seldom satisfied by the great majority of real-life problems. In this paper we propose a new algorithm to deal with fuzzy linear programming problem with fuzzy variables and fuzzy constraints in the requirement vector. We consider the situation where fuzzy numbers are represented by trapezoidal membership functions. A fuzzy optimal solution to the original fuzzy linear programming problem is obtained. An illustrative example is also included to demonstrate our proposed algorithm.

Keywords: Linear Programming problem, Fuzzy Linear Programming Problem, Trapezoidal Fuzzy Numbers.

1. Introduction

A crisp linear programming problem is

$$\begin{aligned} &\max\ Z = c^T x \\ &s.t.\ Ax \le b \\ &x \ge 0 \end{aligned} \tag{1}$$

Where $c = (c_1, c_2, ..., c_n)^T, x = (x_1, x_2, ..., x_n)^T, b = (b_1, b_2, ..., b_m)^T, A = (a_{ij})_{m\times n}, c_j, b_i, a_{ij}$ are all crisp real numbers. In real decision making problems however, it is usual that the coefficients of linear programming where human estimation is used are inexact. In this case fuzzy linear programming problem (FLP) may provide flexibility. A general model for FLP and its solutions has been studied by L Campos, J L Verdegay and M Delgado in [7,8]. A concept of possibilistic linear programming problems with fuzzy constraints is described by Li Xiaozhong in [5]. The variables in these models are considered crisp. Again Li Xiaozhong in [6] has considered fuzzy variables and fuzzy constraints for studying fuzzy linear programming problems (FLP). But he has considered only triangular fuzzy numbers. In the present paper attempt has been made to study the same problem with trapezoidal fuzzy variables and fuzzy constraints.

2. Basic definition

Trapezoidal Fuzzy number: A fuzzy set $\tilde{u}$ on R is said to be a trapezoidal fuzzy number if its membership function is given by,

$$«_{\tilde{u}}(x)=\begin{cases} 0, & \text{if } x \le \underline{u}-\underline{\underline{u}} \text{ and } x \ge \overline{u}+\overline{\overline{u}} \\ (x-\underline{u}+\underline{\underline{u}})/\underline{\underline{u}}, & \text{if } \underline{u}-\underline{\underline{u}} \le x \le \underline{u} \\ (\overline{u}+\overline{\overline{u}}-x)/\overline{\overline{u}}, & \text{if } \overline{u} \le x \le \overline{u}+\overline{\overline{u}} \\ 1, & \text{if } \underline{u} \le x \le \overline{u} \end{cases} \quad (2)$$

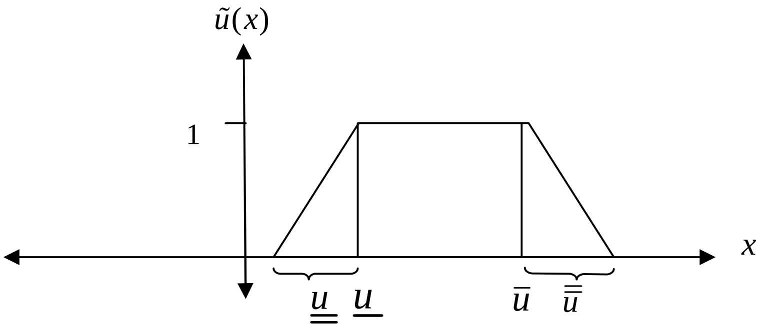

Fig 1: Trapezoidal fuzzy number

Basic definitions and operations of bounded and closed fuzzy numbers can be seen from references.

3. Operations on fuzzy numbers

Let $\tilde{u}=(\underline{u},\overline{u},\underline{\underline{u}},\overline{\overline{u}})_{LR}$ and $\tilde{v}=(\underline{v},\overline{v},\underline{\underline{v}},\overline{\overline{v}})_{LR}$ be two trapezoidal fuzzy numbers and 'k' is any real number, where $\underline{u}$ and $\overline{u}$ are the infimum and supremum of $\{x \in R : \tilde{u}(x)=1\}$ and $\underline{u}-\underline{\underline{u}}$ and $\overline{u}+\overline{\overline{u}}$ are the lower limit and the upper limit of $\{x \in R : \tilde{u}(x)>0\}$. Similarly for $\tilde{v}$. We define the following operations

$\tilde{u}+\tilde{v}=(\underline{u}+\underline{v},\overline{u}+\overline{v},\underline{\underline{u}}+\underline{\underline{v}},\overline{\overline{u}}+\overline{\overline{v}})_{LR}$

$\tilde{u}-\tilde{v}=(\underline{u}-\overline{v},\overline{u}-\underline{v},\underline{\underline{u}}+\underline{\underline{v}},\overline{\overline{u}}+\overline{\overline{v}})_{LR}$

$\tilde{u}\pm k=(\underline{u}\pm k,\overline{u}\pm k,\underline{\underline{u}},\overline{\overline{u}})_{LR}$

$k\tilde{u}=(k\underline{u},k\overline{u},k\underline{\underline{u}},k\overline{\overline{u}})_{LR}$ if $k \ge 0$ and $k\tilde{u}=(k\overline{u},k\underline{u},|k|\underline{\underline{u}},|k|\overline{\overline{u}})_{LR}$ if $k \le 0$

Theorem: If $\tilde{u}=(\underline{u},\overline{u},\underline{\underline{u}},\overline{\overline{u}})_{LR}$ and $\tilde{v}=(\underline{v},\overline{v},\underline{\underline{v}},\overline{\overline{v}})_{LR}$ are two trapezoidal fuzzy numbers then $\tilde{u} \precsim \tilde{v} \Leftrightarrow \underline{u} \le \underline{v}, \overline{u} \le \overline{v}, \underline{u}-\underline{\underline{u}} \le \underline{v}-\underline{\underline{v}}, \overline{u}+\overline{\overline{u}} \le \overline{v}+\overline{\overline{v}}$.

4. Fuzzy Linear Programming Problem

Let $A=(a_{ij})_{m\times n}, c=(c_j)_{n\times 1}, \tilde{b}=(\tilde{b}_i)_{m\times 1}, \tilde{x}=(\tilde{x}_j)_{n\times 1}$ where a_{ij}, c_j are crisp numbers for all $i=1,2,....,m;\ j=1,2,....,n$; $\tilde{b}_i(i=1,...,m)$ are trapezoidal fuzzy numbers and $\tilde{x}_j(j=1,...,n)$ are trapezoidal fuzzy variables. Then

$$\max\ \tilde{Z}=c^T\tilde{x}$$
$$s.t.\ A\tilde{x}\precsim\tilde{b} \qquad (3)$$
$$\tilde{x}\succsim\tilde{0}$$

(3) is said to be a fuzzy linear programming (FLP) problem with fuzzy variables and fuzzy requirement vector.

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5. Algorithm for solving the problem (3)

Step1: Apply the operations defined earlier in the objective functions and constraints. We get

$$\max\ z=c^T\tilde{x}$$
$$=\sum_{j=1}^{n}c_j\tilde{x}_j$$
$$=\sum_{j=1}^{n}c_j(\underline{x}_j,\overline{x}_j,\underline{\underline{x}}_j,\overline{\overline{x}}_j)=(z_1(\underline{x}_1,\overline{x}_1,..,\underline{x}_n,\overline{x}_n),z_2(\underline{x}_1,\overline{x}_1,..,\underline{x}_n,\overline{x}_n),z_3(\underline{\underline{x}}_1,..,\underline{\underline{x}}_n),z_4(\overline{\overline{x}}_1,..,\overline{\overline{x}}_n))$$

$s.t.\ (f_{1i}(\underline{x}_1,\overline{x}_1,..,\underline{x}_n,\overline{x}_n), f_{2i}(\underline{x}_1,\overline{x}_1,..,\underline{x}_n,\overline{x}_n), g_{1i}(\underline{\underline{x}}_1,..,\underline{\underline{x}}_n), g_{2i}(\overline{\overline{x}}_1,..,\overline{\overline{x}}_n))\precsim(\underline{b}_i,\overline{b}_i,\underline{\underline{b}}_i,\overline{\overline{b}}_i)$ for each i=1,2,..,m. Where both $f_{1i}\left(\underline{x}_1,\overline{x}_1,....,\underline{x}_n,\overline{x}_n\right)$ and $f_{2i}\left(\underline{x}_1,\overline{x}_1,....,\underline{x}_n,\overline{x}_n\right)$ are function of $\underline{x}_1,\overline{x}_1,....,\underline{x}_n,\overline{x}_n$ for each $i=1,2,....,m$; $g_{1i}\left(\underline{\underline{x}}_1,\underline{\underline{x}}_2,....,\underline{\underline{x}}_n\right)$ is a function of $\underline{\underline{x}}_1,....,\underline{\underline{x}}_n$ for each $i=1,2,....,m$ and $g_{2i}\left(\overline{\overline{x}}_1,\overline{\overline{x}}_2,....,\overline{\overline{x}}_n\right)$ is a function of $\overline{\overline{x}}_1,....,\overline{\overline{x}}_n$ for each $i=1,2,....,m$.

Step2: Consider the crisp multi-objective problem

$$\max(z_1,z_2)$$
$$s.t.\ f_{1i}\le\underline{b}_i,\ f_{2i}\le\overline{b}_i$$

for each i=1,2,..,m. find its solution $(\underline{x}_j,\overline{x}_j)$ for each j=1,2,..,n.

Step3: Next consider the crisp multi-objective problem

$$\max(z_3,z_4)$$
$$s.t.\ f_{1i}-g_{1i}\le\underline{b}_i-\underline{\underline{b}}_i,\ f_{2i}+g_{2i}\le\overline{b}_i+\overline{\overline{b}}_i,\underline{x}_j-\underline{\underline{x}}_j\ge 0,\overline{x}_j+\overline{\overline{x}}_j\ge 0$$

for each i and j. Find its solution $(\underline{\underline{x}}_j,\overline{\overline{x}}_j)$ for each j=1,2,..,n.

Step4: Solution of the original problem is $\tilde{x}=(\tilde{x}_1,\tilde{x}_2,..,\tilde{x}_n)^T$ where $\tilde{x}_j$ is given by $\tilde{x}_j=(\underline{x}_j,\overline{x}_j,\underline{\underline{x}}_j,\overline{\overline{x}}_j)$.

6. Numerical example

$$
\begin{aligned}
&\max \tilde{Z} = 2\tilde{x}_1 + 3\tilde{x}_2 \\
&s.t. \\
&-3\tilde{x}_1 + \tilde{x}_2 \precsim \tilde{b}_1 \\
&4\tilde{x}_1 + 2\tilde{x}_2 \precsim \tilde{b}_2 \\
&4\tilde{x}_1 - \tilde{x}_2 \precsim \tilde{b}_3 \\
&-\tilde{x}_1 + 2\tilde{x}_2 \precsim \tilde{b}_4 \\
&\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \succsim 0
\end{aligned} \tag{4}
$$

$$
«_{\tilde{b}_1}(x) = \begin{cases} 0, & -1 \geq x, x \geq 3 \\ x+1, & -1 \leq x \leq 0 \\ 1, & 0 \leq x \leq 2 \\ 3-x, & 2 \leq x \leq 3 \end{cases}, \quad «_{\tilde{b}_2}(x) = \begin{cases} 0, & 14 \geq x, x \geq 27 \\ (x-14)/4, & 14 \leq x \leq 18 \\ 1, & 18 \leq x \leq 22 \\ (27-x)/5, & 22 \leq x \leq 27 \end{cases},
$$

$$
«_{\tilde{b}_3}(x) = \begin{cases} 0, & 7 \geq x, x \geq 14 \\ (x-7)/2, & 7 \leq x \leq 9 \\ 1, & 9 \leq x \leq 11 \\ (14-x)/3, & 11 \leq x \leq 14 \end{cases} \quad \text{and} \quad «_{\tilde{b}_4}(x) = \begin{cases} 0, & 3 \geq x, x \geq 8 \\ x-3, & 3 \leq x \leq 4 \\ 1, & 4 \leq x \leq 6 \\ (8-x)/2, & 6 \leq x \leq 8 \end{cases}.
$$

Solution:

*Step*1:

$$
\begin{aligned}
&\max \tilde{Z} = 2(\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1) + 3(\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \\
&s.t. \\
&-3(\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1) + (\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \precsim (0,2,1,1) \\
&4(\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1) + 2(\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \precsim (18,22,4,5) \quad \to \\
&4(\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1) - (\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \precsim (9,11,2,3) \\
&-(\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1) + 2(\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \precsim (4,6,1,2) \\
&\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1, \underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&\max \tilde{Z} = (2\underline{x}_1, 2\overline{x}_1, 2\underline{\underline{x}}_1, 2\overline{\overline{x}}_1) + (3\underline{x}_2, 3\overline{x}_2, 3\underline{\underline{x}}_2, 3\overline{\overline{x}}_2) \\
&s.t. \\
&(-3\overline{x}_1, -3\underline{x}_1, 3\underline{\underline{x}}_1, 3\overline{\overline{x}}_2) + (\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \precsim (0,2,1,1) \\
&(4\underline{x}_1, 4\overline{x}_1, 4\underline{\underline{x}}_1, 4\overline{\overline{x}}_1) + (2\underline{x}_2, 2\overline{x}_2, 2\underline{\underline{x}}_2, 2\overline{\overline{x}}_2) \precsim (18,22,4,5) \to \\
&(4\underline{x}_1, 4\overline{x}_1, 4\underline{\underline{x}}_1, 4\overline{\overline{x}}_1) - (\underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2) \precsim (9,11,2,3) \\
&(-\overline{x}_1, -\underline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1) + (2\underline{x}_2, 2\overline{x}_2, 2\underline{\underline{x}}_2, 2\overline{\overline{x}}_2) \precsim (4,6,1,2) \\
&\underline{x}_1, \overline{x}_1, \underline{\underline{x}}_1, \overline{\overline{x}}_1, \underline{x}_2, \overline{x}_2, \underline{\underline{x}}_2, \overline{\overline{x}}_2 \geq 0
\end{aligned}
$$

$$\max \tilde{Z} = (2\underline{x}_1 + 3\underline{x}_2, 2\bar{x}_1 + 3\bar{x}_2, 2\underline{\underline{x}}_1 + 3\underline{\underline{x}}_2, 2\bar{\bar{x}}_1 + 3\bar{\bar{x}}_2)$$

$$s.t.$$

$$(-3\bar{x}_1 + \underline{x}_2, -3\underline{x}_1 + \bar{x}_2, 3\underline{\underline{x}}_1 + \underline{\underline{x}}_2, 3\bar{\bar{x}}_1 + \bar{\bar{x}}_2) \precsim (0,2,1,1)$$

$$(4\underline{x}_1 + 2\underline{x}_2, 4\bar{x}_1 + 2\bar{x}_2, 4\underline{\underline{x}}_1 + 2\underline{\underline{x}}_2, 4\bar{\bar{x}}_1 + 2\bar{\bar{x}}_2) \precsim (18,22,4,5)$$

$$(4\underline{x}_1 - \bar{x}_2, 4\bar{x}_1 - \underline{x}_2, 4\underline{\underline{x}}_1 + \underline{\underline{x}}_2, 4\bar{\bar{x}}_1 + \bar{\bar{x}}_2) \precsim (9,11,2,3)$$

$$(-\bar{x}_1 + 2\underline{x}_2, -\underline{x}_1 + 2\bar{x}_2, \underline{\underline{x}}_1 + 2\underline{\underline{x}}_2, \bar{\bar{x}}_1 + 2\bar{\bar{x}}_2) \precsim (4,6,1,2)$$

$$\underline{x}_1, \bar{x}_1, \underline{\underline{x}}_1, \bar{\bar{x}}_1, \underline{x}_2, \bar{x}_2, \underline{\underline{x}}_2, \bar{\bar{x}}_2 \geq 0$$

*Step*2:

$$\max Z_{11} = (2\underline{x}_1 + 3\underline{x}_2, 2\bar{x}_1 + 3\bar{x}_2)$$

$$s.t.$$

$$-3\bar{x}_1 + \underline{x}_2 \leq 0$$
$$-3\underline{x}_1 + \bar{x}_2 \leq 2$$
$$4\underline{x}_1 + 2\underline{x}_2 \leq 18$$
$$4\bar{x}_1 + 2\bar{x}_2 \leq 22$$
$$4\underline{x}_1 - \bar{x}_2 \leq 9$$
$$4\bar{x}_1 - \underline{x}_2 \leq 11$$
$$-\bar{x}_1 + 2\underline{x}_2 \leq 4$$
$$-\underline{x}_1 + 2\bar{x}_2 \leq 6$$
$$\underline{x}_1, \bar{x}_1, \underline{x}_2, \bar{x}_2 \geq 0$$

Solving we get $\underline{x}_1 = 2.6667, \underline{x}_2 = 3.6667, \bar{x}_1 = 3.3333, \bar{x}_2 = 4.3333$

*Step*3:

$$\max Z_{12} = (2\underline{\underline{x}}_1 + 3\underline{\underline{x}}_2, 2\bar{\bar{x}}_1 + 3\bar{\bar{x}}_2)$$

$$s.t.$$

$$-3\bar{x}_1 + \underline{x}_2 - 3\underline{\underline{x}}_1 - \underline{\underline{x}}_2 \leq -1$$
$$-3\underline{x}_1 + \bar{x}_2 + 3\bar{\bar{x}}_1 + \bar{\bar{x}}_2 \leq 3$$
$$4\underline{x}_1 + 2\underline{x}_2 - 4\underline{\underline{x}}_1 - 2\underline{\underline{x}}_2 \leq 14$$
$$4\bar{x}_1 + 2\bar{x}_2 + 4\bar{\bar{x}}_1 + 2\bar{\bar{x}}_2 \leq 27$$
$$4\underline{x}_1 - \bar{x}_2 - 4\underline{\underline{x}}_1 - \underline{\underline{x}}_2 \leq 7$$
$$4\bar{x}_1 - \underline{x}_2 + 4\bar{\bar{x}}_1 + \bar{\bar{x}}_2 \leq 14$$
$$-\bar{x}_1 + 2\underline{x}_2 - \underline{\underline{x}}_1 - 2\underline{\underline{x}}_2 \leq 3$$
$$-\underline{x}_1 + 2\bar{x}_2 + \bar{\bar{x}}_1 + 2\bar{\bar{x}}_2 \leq 8$$
$$\underline{x}_1 - \underline{\underline{x}}_1 \geq 0, \bar{x}_1 + \bar{\bar{x}}_1 \geq 0, \underline{x}_2 - \underline{\underline{x}}_2 \geq 0, \bar{x}_2 + \bar{\bar{x}}_2 \geq 0$$
$$\underline{\underline{x}}_1, \bar{\bar{x}}_1, \underline{\underline{x}}_2, \bar{\bar{x}}_2 \geq 0$$

Solving this problem we get $\underline{\underline{x}}_1 = 2.6667, \underline{\underline{x}}_2 = 3.6667, \bar{\bar{x}}_1 = 0.9524, \bar{\bar{x}}_2 = 0.5238$

*Step*4: Finally we have solution of the original problem as follows $\tilde{x}_1 = (2.6667, 3.3333, 2.6667, 0.9524), \tilde{x}_2 = (3.6667, 4.3333, 3.6667, 0.5238)$

7. Conclusion

In this paper we have discussed a general method for solving a FLP with trapezoidal fuzzy variables and fuzzy constraints with b_i trapezoidal fuzzy numbers.

Foot Note: If the resource vectors in the constraints of the original problem are defuzzified then the solution of the problem becomes $x_1 = 3.0000,\ x_2 = 4.0833\ \textit{and}\ Z = 18.2499$ and the defuzzification of the solution obtained by our method gives $x_1 = 2.7143,\ x_2 = 3.4762\ \textit{and}\ Z = 15.8572$.

8. References

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