On Fuzzy Real-valued Bounded Variation Multiple Sequence Space $\overline{bv}^F(p)$

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Abstract: The idea of difference sequence spaces (for single sequences) was first introduced by Kizmaz in 1981 and the idea of triple sequences was first introduced by Sahiner et.al. in 2007. The purpose of this paper is to extend a generalized convergence method to sequence $bv(p)$ of fuzzy numbers of multiplicity greater than two. Here we introduce the notion of fuzzy real-valued bounded variation difference multiple sequences $\overline{bv}^F(p)$ where $p = \{p_{nk}\}$ is a triple sequence of bounded strictly positive numbers. We study different algebraic and topological properties of the space like completeness, solid, monotone, symmetric, convergence free etc. We prove some inclusion results too.

Keywords: Fuzzy real valued triple sequence, multiple sequences, bounded variation, solid, monotone, symmetric, convergence free, sequence algebra.

INTRODUCTION

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh [23] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. In fact the fuzzy set theory has become an area of active area of research in science and engineering for the last 40 years. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It extends the scope and results of classical mathematical analysis by applying fuzzy logic to conventional mathematical objects, such as functions, sequences and series etc. While studying fuzzy topological spaces, we face many situations where we need to deal with convergence of fuzzy numbers.

Using the notion of fuzzy real numbers, different types of fuzzy real-valued sequence spaces have been introduced and studied by several mathematicians. The initial works on double sequences of real or complex terms are found in Bromwich [2]. Hardy [8] introduced the notion of regular convergence for double sequences of real or complex terms. Agnew [1] studied the summability theory...
of multiple sequences and obtained certain theorems which have already been proved for double sequences by the author himself. Móricz [14] extended statistical convergence from single to multiple real sequences and obtained some results for real double sequences. Matloka [13] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. There are many applications of the sequences and difference sequences of numbers (real, complex and fuzzy numbers). For example sequences of numbers have unexpected and practical uses in many areas of science and radar echoes from planets, the travel times of deep-ocean sound waves for monitoring ocean temperature, and improving synthetic speech and the sounds associated with computer music. Furthermore, it is shown by Kawamura et. al. [10] that the earthquake ground motions have very simple conditioned fuzzy set rules with non-fuzzy parameters of the first and second order differences $\Delta x_k$ and $\Delta^2 x_k$ defined by membership functions. Therefore the difference sequences of fuzzy numbers are used, for example in the prediction of earthquake waves.

II. DEFINITIONS AND PRELIMINARIES

In this section, we recall some notations and basic definitions which will be used in this paper. Throughout the paper $N, R$ and $C$ denote the sets of natural, real and complex numbers respectively and $c, c_0, \ell_\infty$ denote the spaces of convergent, null and bounded sequences respectively.

Let $C(R^n) = \{A \subset R^n : A$ is compact and convex}. The space $C(R^n)$ has a linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between $A$ and $B$ of $C(R^n)$ is defined as

$$\delta_\infty (A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|; \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}.$$ 

Then $(C(R^n), \delta_\infty)$ is a complete metric space.

A fuzzy real number $X$ is a fuzzy set on $R$, i.e. a mapping $X : R \rightarrow L([0,1])$ associating each real number $t$ with its grade of membership $X(t)$. Every real number $r$ can be expressed as a fuzzy real number $\tilde{r}$ as follows:

$$\tilde{r}(t) = \begin{cases} 1 & \text{if } t = r \\ 0 & \text{otherwise} \end{cases}$$

The $\alpha$-level set of a fuzzy real number $X$, $0 < \alpha \leq 1$ denoted by $[X]_\alpha$ is defined as

$$[X]_\alpha = \{t \in R : X(t) \geq \alpha\}.$$ 

A fuzzy real number $X$ is called convex if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called normal. A fuzzy real number $X$ is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}[0, a + \varepsilon)$, for all $a \in L$ is open in the usual topology of $R$. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$.

The Arithmetic operations on $R(L)$ are defined as follows:

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If $X,Y \in R(L)$, then
\[
(X \Theta Y)(t) = \sup_{s \in R} \{X(s) \land Y(s-t)\}, \quad t \in R,
\]
\[
(X \oplus Y)(t) = \sup_{s \in R} \{X(s) \lor Y(s-t)\}, \quad t \in R,
\]
\[
(X \otimes Y)(t) = \sup_{s \in R} \{X(s) \land Y(t/s)\}, \quad t \in R,
\]
\[
(X / Y)(t) = \sup_{s \in R} \{X(st) \land Y(s)\}, \quad t \in R.
\]

The absolute value $|X|$ of $X \in R(L)$ is defined as
\[
|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}
\]

Let $D$ be the set of all closed bounded intervals $X = [X^l, X^r]$ on the real line $R$. Then $X \leq Y$ if and only if $X^l \leq Y^l$ and $X^r \leq Y^r$. Also let $d(X,Y) = \max(|X^l - X^r|, |Y^l - Y^r|)$. Then $(D,d)$ is a complete metric space.

Let $\overline{d} : R(L) \times R(L) \to R$ be defined by $\overline{d}(X,Y) = \sup_{0 \leq x \leq 1} d([X]^x,[Y]^x)$, for $X,Y \in R(L)$. Then $\overline{d}$ defines a metric on $R(L)$ and $(R(L),\overline{d})$ is a complete metric space.

A triple sequence (real or complex) can be defined as a function $x : N \times N \times N \to R(C)$.

The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner et. al. [17], Dutta et. al. [4], P. Kumar et. al. [12] and many others. Recently Savas and Esi [18] have introduced statistical convergence of triple sequences on probabilistic normed space. Later on, Esi [6] have introduced statistical convergence of triple sequences in topological groups. Some more works on triple sequences are found in [3, 5].

A fuzzy real-valued triple sequence $X = \langle X_{n,k} \rangle$ is a triple infinite array of fuzzy real numbers $X_{n,k}$ for all $n, l, k \in N$ and is denoted by $\{X_{n,k}\}$. where $X_{n,k} \in R(L)$.

A fuzzy real-valued triple sequence $X = \langle X_{n,k} \rangle$ is said to be convergent in Pringsheims sense to the fuzzy real number $X$, if for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon), l_0 = l_0(\varepsilon), k_0 = k_0(\varepsilon) \in N$ such that $\overline{d}(X_{n,k}, X) < \varepsilon$ for all $n \geq n_0, l \geq l_0, k \geq k_0$.

A fuzzy real-valued triple sequence $X = \langle X_{n,k} \rangle$ is said to be a Cauchy sequence, if for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that $\overline{d}(X_{n,k}, X_{r,q}) < \varepsilon$ for every $n \geq p \geq n_0, l \geq q \geq l_0, k \geq r \geq k_0$.

A fuzzy real-valued triple sequence $X = \langle X_{n,k} \rangle$ is said to be bounded if there exists a positive integer $M$ such that $\overline{d}(X_{n,k}, \overline{0}) < M$ for all $n, l, k$.

Let $\ell_\infty$ denote the set of all bounded triple sequences of fuzzy numbers which is a normed space, normed by $\|X\| = \sup_{n,k,l} |X_{n,k}|$. 

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A fuzzy real-valued triple sequence space $E^F$ is said to be solid if $\{Y_{nk}\} \in E^F$ whenever $|y_{nk}| \leq |X_{nk}|$ for all $n,l,k \in \mathbb{N}$ and $\{X_{nk}\} \in E^F$.

A fuzzy real-valued triple sequence space $E^F$ is said to be monotone if $E^F$ contains the canonical pre-image of all its step spaces.

A fuzzy real-valued triple sequence $E^F$ is said to be symmetric if $S(X) \subset E^F$ for all $X \in E^F$, where $S(X)$ denotes the set of all permutations of the elements of $X = \{X_{nk}\}$.

A fuzzy real-valued triple sequence space $E^F$ is said to be sequence algebra if $\{X_{nk} \otimes Y_{nk}\} \in E^F$, whenever $\{X_{nk}\}, \{Y_{nk}\} \in E^F$.

A fuzzy real-valued triple sequence space $E^F$ is said to be convergence free if $\{Y_{nk}\} \in E^F$, whenever $\{X_{nk}\} \in E^F$ and $X_{nk} = \bar{0}$ implies $Y_{nk} = \bar{0}$.

The notion of difference sequence spaces was introduced by Kizmaz [11] as follows:

$$Z(\Delta) = \{x = (x_k) \in w: (\Delta x_k) \in Z\}, \text{ for } Z = \ell_p, c, c_0, \text{ where } \Delta x_k = x_k - x_{k+1}, \text{ for all } k \in \mathbb{N}.$$

Tripathy and Esi[21] introduced the notion of difference sequence space $\Delta m x = (\Delta m x_k) = x_k - x_{k+m}$, for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$ be fixed.

Later on it was studied by Tripathy and Sarma [22] and introduced difference double sequence spaces as follows:

$$Z(\Delta) = \{x = (X_{nk}) \in 2^w: (\Delta X_{nk}) \in Z\}, \text{ for } Z = 2^\ell_p, 2c, 2c_0, \text{ where } \Delta X_{nk} = X_{n,k} - X_{n,k+1} - X_{n+1,k} + X_{n+1,k+1}, \text{ for all } n,k \in \mathbb{N}.$$

For fuzzy triple sequences, we introduce the following:

$$\Delta X_{nk} = X_{nk} - X_{n,l+1,k} - X_{n,l+1,k+1} + X_{n+l+1,k} - X_{n+l+1,k+1} + X_{n+1,l+1,k} - X_{n+1,l+1,k+1}, \text{ for all } n,l,k \in \mathbb{N} \quad (1)$$

The properties of bounded variation sequences have been applied for investigations in various branches of science and engineering. Some works on bounded variation sequences are found in [7, 15, 19, 20].

Let $p = \{p_{nk}\}$ be a triple sequence of bounded strictly positive numbers. We introduce the following fuzzy real-valued bounded variation triple sequence space $3 \text{bv}^F(p)$ as follows:

$$3 \text{bv}^F(p) = \left\{ X = \{X_{nk}\} : \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} [d(\Delta X_{nk}, \Theta)]_{p_{nk}} < \infty \right\}.$$

where $\Delta X_{nk} = X_{nk} - X_{n,l+1,k} - X_{n,l+1,k+1} + X_{n+l+1,k} - X_{n+l+1,k+1} + X_{n+1,l+1,k} - X_{n+1,l+1,k+1}$ for all $n,l,k \in \mathbb{N}$.

For establishing the results of this chapter, we procure the following existing result.

**Lemma.** If a sequence space $E^F$ is solid, then it is monotone.
III. MAIN RESULTS

Theorem 1. The class of sequences \( bv^F(p) \) is a complete metric space with respect to the metric \( \rho \) defined by

\[
\rho(X,Y) = \sup_{n,l} \overline{d}(X_{nl}, Y_{nl}) + \sup_{n,k} \overline{d}(X_{nk}, Y_{nk}) + \sup_{l,k} \overline{d}(X_{lk}, Y_{lk})
\]

\[
+ \left( \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ \overline{d}^a(\Delta X_{nl}, \Delta Y_{nl}) \right] \right)^{\frac{1}{M}}, \quad \text{where } M = \max (1, \sup_{n,l,k} p_{nk}).
\]

Proof. Let \( \{X^{(i)}\} \) be a Cauchy sequence in \( bv^F(p) \), where \( X^{(i)} = \{X_{nk}^{(i)}\} \).

Then for a given \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that

\[
\rho(X^{(i)}, X^{(j)}) < \varepsilon, \quad \text{for all } i, j \geq n_0 \quad (2)
\]

\[
\Rightarrow \sup_{n,l} \overline{d}(X_{nl}^{(i)}, X_{nl}^{(j)}) + \sup_{n,k} \overline{d}(X_{nk}^{(i)}, X_{nk}^{(j)}) + \sup_{l,k} \overline{d}(X_{lk}^{(i)}, X_{lk}^{(j)})
\]

\[
+ \left( \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ \overline{d}(\Delta X_{nk}, \Delta Y_{nk}) \right] \right)^{\frac{1}{M}} < \varepsilon, \quad \text{for all } i, j \geq n_0 \quad (3)
\]

From (3), we obtain for each fixed \( n, l, k \in \mathbb{N} \),

\[
\overline{d}(X_{nl}^{(i)}, X_{nl}^{(j)}) < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq n_0
\]

\[
\overline{d}(X_{nk}^{(i)}, X_{nk}^{(j)}) < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq n_0
\]

\[
\overline{d}(X_{lk}^{(i)}, X_{lk}^{(j)}) < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq n_0
\]

and \( \overline{d}(\Delta X_{nk}^{(i)}, \Delta X_{nk}^{(j)}) < \frac{\varepsilon}{3}, \quad \text{for all } i, j \geq n_0 \).

This implies that

\( \{X_{nl}^{(i)}\} \) is a Cauchy sequence in \( R(L) \) for each \( n, l \in \mathbb{N} \),

\( \{X_{nk}^{(i)}\} \) is a Cauchy sequence in \( R(L) \) for each \( n, k \in \mathbb{N} \)

\( \{X_{lk}^{(i)}\} \) is a Cauchy sequence in \( R(L) \) for each \( l, k \in \mathbb{N} \) and

\( \{\Delta X_{nk}^{(i)}\} \) is a Cauchy sequence in \( R(L) \) for each \( n, l, k \in \mathbb{N} \).

Since \( R(L) \) is complete, so \( \{X_{nl}^{(i)}\}, \{X_{nk}^{(i)}\}, \{X_{lk}^{(i)}\} \) and \( \{\Delta X_{nk}^{(i)}\} \) are convergent for each \( n, l, k \in \mathbb{N} \).
Let \( \lim_{i \to \infty} X_{n1}^{(i)} = X_{n1} \), for each \( n, l \in \mathbb{N} \). Also let \( \lim_{i \to \infty} X_{nlk}^{(i)} = X_{nlk} \), for each \( n, k \in \mathbb{N} \) and \( \lim_{i \to \infty} X_{1lk}^{(i)} = X_{1lk} \), for each \( l, k \in \mathbb{N} \).

Let us consider \( \Delta X_{111}^{(i)} \).

Now \( \Delta X_{111} = X_{111} - X_{121} - X_{121} + X_{212} - X_{211} + X_{212} - X_{222} \).

Then \( \langle X_{111}^{(i)} \rangle, \langle X_{121}^{(i)} \rangle, \langle X_{121}^{(i)} \rangle, \langle X_{122}^{(i)} \rangle, \langle X_{211}^{(i)} \rangle, \langle X_{212}^{(i)} \rangle \) and \( \langle X_{222}^{(i)} \rangle \) are convergent.

Hence \( \langle X_{222}^{(i)} \rangle \) converges.

Let \( \lim_{i \to \infty} X_{222}^{(i)} = X_{222} \).

Proceeding in this way inductively, we have \( \langle X_{n1k}^{(i)} \rangle \) converges for each \( n, l \in \mathbb{N} \).

Let \( \lim_{i \to \infty} X_{n1k}^{(i)} = X_{n1k} \), for all \( n, l, k \in \mathbb{N} \), where \( X = \langle X_{n1k} \rangle \).

Taking limit as \( j \to \infty \) in equation (2), we have

\( \rho \left( X^{(i)} , X \right) < \epsilon \), for all \( i \geq n_0 \).

So the Cauchy sequence \( \langle X^{(i)} \rangle \) converges to \( X \).

Now for all \( i \geq n_0 \),

\( \rho \left( X, \overline{0} \right) \leq \rho \left( X, X^{(i)} \right) + \rho \left( X^{(i)}, \overline{0} \right) \leq \epsilon + K < \infty. \)

This gives \( X \in \mathcal{L}^F(p) \) and so the space \( \mathcal{L}^F(p) \) is complete.

**Theorem 2.** The class of sequences \( \mathcal{L}^F(p) \) is neither solid nor monotone.

**Proof.** The result follows from the following example

**Example 1.** Let \( p_{nk} = \begin{cases} 2, & \text{if } n = l = k \\ 1, & \text{otherwise} \end{cases} \)

Consider the sequence \( \langle X_{nk} \rangle \) defined by:

\[
X_{1lk} = \overline{1}, \quad \text{for all } l, k \geq 2
\]

\[
X_{n1k} = -\overline{1}, \quad \text{for all } n, k \geq 2
\]

\[
X_{nl1} = \overline{0}, \quad \text{for all } n, l \geq 2,
\]

\[
X_{nnn}(t) = \begin{cases} 1 + 3n^3 t, & \text{for } -\frac{1}{3n^3} \leq t \leq 0 \\ 1 - 3n^3 t, & \text{for } 0 \leq t \leq \frac{1}{3n^3} \\ 0, & \text{otherwise} \end{cases}
\]

and \( X_{nk} = \overline{0}, \text{ otherwise} \)

Then \( \langle \Delta X_{nk} \rangle \) is given by

For \( n = l = k \),
\[ \Delta X_{nnn}(t) = \begin{cases} 
\left[ 1 + \frac{3n^3(n+1)^3}{\binom{n}{2} + 3n + 1} \right] t, & \text{for } -\frac{3n^2 + 3n + 1}{3n^3(n+1)^3} \leq t \leq 0 \\
\left[ 1 - \frac{3n(n+1)}{3n^2 + 3n + 1} \right] t, & \text{for } 0 \leq t \leq \frac{3n^2 + 3n + 1}{3n^3(n+1)^3} \\
0, & \text{otherwise}
\end{cases} \]

and

\[ \Delta X_{nik} = \begin{cases} 
-X_{nxx} \forall n, l, k \text{ with } n = l + 1 = k \text{ and } n = l = k + 1 \\
X_{nxx} \forall n, l, k \text{ with } n - 1 = l = k \\
-X_{ill} \forall n, l, k \text{ with } n + 1 = l = k \\
X_{ill} \forall n, l, k \text{ with } n + l + 1 = k \\
X_{kkk} \forall n, l, k \text{ with } n = l = k - 1
\end{cases} \]

and \( \Delta X_{nik} = \tilde{0} \), otherwise.

Then we have

\[ \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ \overrightarrow{d}(X_{nik}, 0) \right]_{p_{nk}} = 3 \sum_{n=1}^{\infty} \left[ \overrightarrow{d}(X_{nxx}, 0) \right]_{p_{nk}} + 2 \sum_{l=1}^{\infty} \left[ \overrightarrow{d}(X_{ill}, 0) \right]_{p_{nk}} + \sum_{k=1}^{\infty} \left[ \overrightarrow{d}(X_{kkk}, 0) \right]_{p_{nk}} \]

\[ + \sum_{n=1}^{\infty} \left[ \overrightarrow{d}(X_{nxx} - X_{(n+l)(n+l)(n+l)}, 0) \right]_{p_{nk}} \]

\[ = 3 \sum_{n=1}^{\infty} \frac{1}{3} + 2 \sum_{l=1}^{\infty} \frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{3} + \sum_{n=1}^{\infty} \left( \frac{1}{3n^3} - \frac{1}{3(n+1)^3} \right)^2 < \infty. \]

Hence \( \{ X_{nik} \} \in \text{b}v^F(p) \).

Now we consider the sequence \( \{ Y_{nk} \} \) defined by:

\[ Y_{lkk} = 1, \text{ for all } l, k \text{ even} \]
\[ Y_{nkk} = -1, \text{ for all } n, k \text{ even} \]
\[ Y_{nll} = 0, \text{ for all } n, l \text{ even} \]
\[ Y_{nkl} = \tilde{0}, \text{ otherwise.} \]

Then clearly

\[ \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ \overrightarrow{d}(Y_{nik}, 0) \right]_{p_{nk}} = \infty. \]

So \( \{ Y_{nk} \} \notin \text{b}v^F(p) \), and \( \overrightarrow{d}(Y_{nk}, 0) \leq \overrightarrow{d}(X_{nk}, 0) \) for all \( n, l, k \in N \).

Hence the class of sequences \( \text{b}v^F(p) \) is not solid.

So \( \text{b}v^F(p) \) is not monotone by Lemma.

**Theorem 3.** The class of sequences \( \text{b}v^F(p) \) is not convergence free in general.

**Proof.** The result follows from the following example.

**Example 2.** Let \( p_{nk} = \begin{cases} 
2, & \text{if } n = l = k \\
1, & \text{otherwise}
\end{cases} \)

We define the sequence \( \{ X_{nk} \} \) defined by:
\[ X_{nnn}(t) = \begin{cases} 
1 + 2n^2t, & \text{for } -\frac{1}{2n^2} \leq t \leq 0 \\
1 - 2n^2t, & \text{for } 0 \leq t \leq \frac{1}{2n^2} \\
0, & \text{otherwise} 
\end{cases} \]

and \( X_{nlk} = \tilde{0} \), for \( n \neq l \neq k \).

Then \( \langle \Delta X_{nlk} \rangle \) is given by

For \( n = l = k \),

\[ \Delta X_{nnn}(t) = \begin{cases} \left(1 + \frac{2n^2(n+1)^2}{2n+1}t\right), & \text{for } -\frac{2n+1}{2n^2(n+1)^2} \leq t \leq 0 \\
\left(1 - \frac{2n^2(n+1)^2}{2n+1}t\right), & \text{for } 0 \leq t \leq \frac{2n+1}{2n^2(n+1)^2} \\
0, & \text{otherwise} \end{cases} \]

and \[ \Delta X_{nlk} = \begin{cases} -X_{nnn} & \forall n, l, k \text{ with } n = l + 1 = k \text{ and } n = l = k + 1 \\
X_{nnn} & \forall n, l, k \text{ with } n-1 = l = k \\
-X_{lll} & \forall n, l, k \text{ with } n+1 = l = k \\
X_{lll} & \forall n, l, k \text{ with } n = l - 1 = k \\
X_{kkk} & \forall n, l, k \text{ with } n = l = k - 1 \end{cases} \]

and \( \Delta X_{nlk} = \tilde{0} \), otherwise.

Then we have

\[
\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ \tilde{d}(\Delta X_{nlk}, \tilde{0}) \right]_{n=l=k}^{p} = 3 \sum_{n=1}^{\infty} \left[ \tilde{d}(X_{nnn}, \tilde{0}) \right]_{n=l=k}^{p} + 2 \sum_{l=1}^{\infty} \left[ \tilde{d}(X_{lll}, \tilde{0}) \right]_{n=l=k}^{p} + \sum_{k=1}^{\infty} \left[ \tilde{d}(X_{kkk}, \tilde{0}) \right]_{n=l=k}^{p} + \sum_{n=1}^{\infty} \left[ \tilde{d}(X_{nnn} - X_{(n+1)(n+1)(n+1)}, \tilde{0}) \right]_{n=l=k}^{p} \\
= 3 \sum_{n=1}^{\infty} \frac{1}{2} + 2 \sum_{l=1}^{\infty} \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{1}{2n^2} - \frac{1}{2(n+1)^2} \right)^2 < \infty. \\
\text{Hence} \ \{ X_{n} \}_{i=1}^{n} \in \text{bn}^{p} \ (p). \\
\]

Let us consider the sequence \( \langle Y_{nlk} \rangle \) defined as follows:

\[ Y_{nnn}(t) = \begin{cases} 
1 + 2nt, & \text{for } -\frac{1}{2n} \leq t \leq 0 \\
1 - 2nt, & \text{for } 0 \leq t \leq \frac{1}{2n} \\
0, & \text{otherwise} 
\end{cases} \]

And \( Y_{nlk} = \tilde{0} \), for \( n \neq l \neq k \).

Then \( \langle \Delta Y_{nlk} \rangle \) is given by
For $n = l = k$, 
\[
\Delta Y_{nn}(t) = \begin{cases} 
1 + 2n(n + 1)t, & \text{for } -\frac{1}{2n(n + 1)} \leq t \leq 0 \\
1 - 2n(n + 1)t, & \text{for } 0 \leq t \leq \frac{1}{2n(n + 1)} \\
0, & \text{otherwise}
\end{cases}
\]

and
\[
\Delta Y_{nk} = \begin{cases} 
-Y_{nn} \forall n, l, k \text{ with } n = l + 1 = k \text{ and } n = l = k + 1 \\
Y_{nn} \forall n, l, k \text{ with } n - 1 = l = k \\
-Y_{ww} \forall n, l, k \text{ with } n + 1 = l = k \\
Y_{ww} \forall n, l, k \text{ with } n = l - 1 = k \\
Y_{kk} \forall n, l, k \text{ with } n = l = k - 1 \\
0, & \text{otherwise.}
\end{cases}
\]

and $\Delta Y_{nk} = 0$, otherwise.

Then we have
\[
\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta Y_{nk}, \bar{0}) \right]^{p_n} = 3 \sum_{n=1}^{\infty} \left[ d(Y_{nn}, \bar{0}) \right]^{p_n} + 2 \sum_{l=1}^{\infty} \left[ d(Y_{ll}, \bar{0}) \right]^{p_n} + \sum_{k=1}^{\infty} \left[ d(Y_{kk}, \bar{0}) \right]^{p_n} + \sum_{n=1}^{\infty} \left[ d(Y_{nn} - Y_{(n+1)(n+1)}, \bar{0}) \right]^{p_n}
\]
\[
= 3 \sum_{n=1}^{\infty} \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \sum_{k=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{2(n + 1)} \right)^2 \right) = \infty.
\]

So $\langle Y_{nk} \rangle \notin \mathbb{b} v^F (p)$.

Hence the classes of sequences $\mathbb{b} v^F (p)$ is not convergence free.

**Theorem 4.** The class of sequences $\mathbb{b} v^F (p)$ is not symmetric.

**Proof.** The result follows from the following example.

**Example 3.** Let $p_{nk} = \begin{cases} 
2, & \text{if } n = l = k \\
1, & \text{otherwise}
\end{cases}$

Consider the sequence $\langle X_{nk} \rangle$ defined by:

$X_{1k} = \bar{1}$, for all $l, k \geq 2$

$X_{nk} = \bar{1}$, for all $n, k \geq 2$

$X_{nn} = \bar{0}$, for all $n, l \geq 2$.

$X_{nn}(t) = \begin{cases} 
1 + 3n^3 t, & \text{for } -\frac{1}{3n^3} \leq t \leq 0 \\
1 - 3n^3 t, & \text{for } 0 \leq t \leq \frac{1}{3n^3} \\
0, & \text{otherwise}
\end{cases}
\]

and $X_{nk} = \bar{0}$, otherwise

Then $\langle \Delta X_{nk} \rangle$ is given by

For $n = l = k$, 

\[
\Delta X_{nnn}(t) = \begin{cases} 
1 + \frac{3n^3(n+1)^3}{3n^2+3n+1} & , \text{for } -\frac{3n^2+3n+1}{3n^3(n+1)^3} \leq t \leq 0 \\
1 - \frac{3n^2(n+1)^3}{3n^2+3n+1} & , \text{for } 0 \leq t \leq \frac{3n^2+3n+1}{3n^3(n+1)^3} \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\Delta X_{nlk} = \begin{cases} 
- X_{nnn} & \forall n, l, k \text{ with } n = l + 1 = k \text{ and } n = l = k + 1 \\
X_{nnn} & \forall n, l, k \text{ with } n - 1 = l = k \\
- X_{lll} & \forall n, l, k \text{ with } n + 1 = l = k \\
X_{lll} & \forall n, l, k \text{ with } n - 1 = l = k \\
X_{kkk} & \forall n, l, k \text{ with } n = l = k - 1
\end{cases}
\]

and \( \Delta X_{nlk} = 0 \), otherwise

Then we have

\[
\sum_{n=3}^{\infty} \sum_{l=3}^{\infty} \sum_{k=3}^{\infty} \left[ d(\Delta X_{nlk}, \tilde{0}) \right] = 3 \sum_{n=3}^{\infty} \left[ d(X_{nnn}, \tilde{0}) \right] + 2 \sum_{l=3}^{\infty} \left[ d(X_{lll}, \tilde{0}) \right] + \sum_{k=3}^{\infty} \left[ d(X_{kkk}, \tilde{0}) \right] + \sum_{n=3}^{\infty} \left[ d(X_{nnn} - X_{(n+1)(n+1)(n+1)}, \tilde{0}) \right] < \infty.
\]

Hence \( \{ X_{nlk} \} \in bV \) (p).

Let us consider the sequence \( \{ Y_{nlk} \} \), a rearrangement of \( \{ X_{nlk} \} \) defined as

\[
Y_{nnn} = X_{nnn},
\]

For \( n \neq l \neq k \),

If \( n \) is even, \( Y_{nlk} = \tilde{0} \)

If \( n \) is odd,

\[
Y_{nlk} = \begin{cases} 
\tilde{1} & \text{if } l, k \text{ are both even} \\
- \tilde{1} & \text{if } l, k \text{ are both odd} \\
0 & \text{otherwise}
\end{cases}
\]

Then

\[
\Delta Y_{nnn} = \begin{cases} 
X_{nnn} - X_{(n+1)(n+1)(n+1)} + \tilde{1} & \text{when } n \text{ is odd} \\
X_{nnn} - X_{(n+1)(n+1)(n+1)} - \tilde{1} & \text{when } n \text{ is even}
\end{cases}
\]

and

\[
\Delta Y_{nlk} = \begin{cases} 
- Y_{nnn} & \forall n, l, k \text{ with } n = l + 1 = k \text{ and } n = l = k + 1 \\
Y_{nnn} & \forall n, l, k \text{ with } n - 1 = l = k \\
- Y_{lll} & \forall n, l, k \text{ with } n + 1 = l = k \\
Y_{lll} & \forall n, l, k \text{ with } n - 1 = l = k \\
Y_{kkk} & \forall n, l, k \text{ with } n = l = k - 1
\end{cases}
\]

and \( \Delta Y_{nlk} = 0 \) otherwise
Then clearly \( \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta Y_{nk}, 0) \right]^{p_{nk}} = \infty \).

This implies \( \langle Y_{nk} \rangle \not\in \_3 \text{bv}^F (p) \) and so the class of sequences \( \_3 \text{bv}^F (p) \) is not symmetric.

**Theorem 5.** The class of sequences \( \_3 \text{bv}^F (p) \) is a sequence algebra.

**Proof.** Let \( \langle X_{nk} \rangle, \langle Y_{nk} \rangle \in \_3 \text{bv}^F (p). \)

\[
\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk}, 0) \right]^{p_{nl}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta Y_{nk}, 0) \right]^{p_{nl}} < \infty
\]

Now \( \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk} \otimes \Delta Y_{nk}, \theta) \right]^{p_{nl}} < \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk}, \theta) \right]^{p_{nl}} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta Y_{nk}, \theta) \right]^{p_{nl}} < \infty \).

This implies \( \langle X_{nk} \otimes Y_{nk} \rangle \in \_3 \text{bv}^F (p) \). Hence \( \_3 \text{bv}^F (p) \) is a sequence algebra.

**Theorem 6.** If \( 0 < p_{nk} < q_{nk} \leq \sup q_{nk}, \) then \( \_3 \text{bv}^F (p) \subset \_3 \text{bv}^F (q) \) and the inclusion is proper.

**Proof.** Let \( \langle X_{nk} \rangle \in \_3 \text{bv}^F (p). \)

Then \( \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk}, 0) \right]^{p_{nk}} < \infty \)

So there exists \( n_0, l_0, k_0 \in \mathbb{N} \) such that

\[
\left[ d(\Delta X_{nk}, 0) \right]^{p_{nk}} < 1, \text{ for } n \geq n_0 \text{ or } l \geq l_0 \text{ or } k \geq k_0 \text{ or for all}
\]

\[
\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk}, 0) \right]^{p_{nk}} < \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk}, 0) \right]^{p_{nk}} \text{ for all } n \geq n_0, l \geq l_0, k \geq k_0.
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ d(\Delta X_{nk}, 0) \right]^{p_{nk}} < \infty.
\]

Thus \( \langle X_{nk} \rangle \in \_3 \text{bv}^F (q). \)

Hence \( \_3 \text{bv}^F (p) \subset \_3 \text{bv}^F (q). \)

To prove the inclusion to be proper, we consider the following example.

**Example 4.** Let \( p_{nk} = \begin{cases} 3, & \text{if } n = l = k \\ 1, & \text{otherwise} \end{cases} \)

and \( q_{nk} = \begin{cases} 4 + \frac{1}{n}, & \text{if } n = l = k \\ 5, & \text{otherwise} \end{cases} \)

We consider the sequence \( \langle X_{nk} \rangle \) defined as follows:

For \( n = l = k \),

\[
X_{n+n} (t) = \begin{cases} 1 + 3nt, & \text{for } -\frac{1}{3n} \leq t \leq 0 \\ 1 - 3nt, & \text{for } 0 \leq t \leq \frac{1}{3n} \\ 0, & \text{otherwise} \end{cases}
\]

and \( X_{nk} = \bar{0}, \text{otherwise} \).
Then the sequence \( \Delta X_{n\ell k} \) is given by

\[
\Delta X_{n\ell k}(t) = \begin{cases} 
(1 + 3n(n + 1)t), & \text{for } -\frac{1}{3n(n + 1)} \leq t \leq 0 \\
(1 - 3n(n + 1)t), & \text{for } 0 \leq t \leq \frac{1}{3n(n + 1)} \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\Delta X_{n\ell k} = \begin{cases} 
-X_{n\ell k} & \forall n, \ell, k \text{ with } n + 1 = k \text{ and } n = k + 1 \\
X_{n\ell k} & \forall n, \ell, k \text{ with } n - 1 = k \\
-X_{n\ell k} & \forall n, \ell, k \text{ with } n + 1 = k \\
X_{n\ell k} & \forall n, \ell, k \text{ with } n = k - 1 \\
X_{n\ell k} & \forall n, \ell, k \text{ with } n = k = 1
\end{cases}
\]

\[
\Delta X_{n\ell k} = \mathfrak{d} \text{ otherwise}.
\]

Then we have

\[
\sum_{n=1}^{\infty} \sum_{l=1}^{3n} \left[ d(X_{n\ell k}, 0) \right]^{a} = 3 \sum_{n=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{a} + 2 \sum_{k=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{a} + \sum_{n=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{a} + \sum_{n=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{a} < \infty.
\]

Hence \( \{X_{n\ell k}\} \in \mathcal{bV}^{\mathcal{A}}(q) \).

and

\[
\sum_{n=1}^{\infty} \sum_{l=1}^{3n} \left[ d(X_{n\ell k}, 0) \right]^{p} = 3 \sum_{a=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{p} + 2 \sum_{b=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{p} + \sum_{c=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{p} + \sum_{d=1}^{\infty} \left[ d(X_{n\ell k}, 0) \right]^{p} = \infty.
\]

Hence \( \{X_{n\ell k}\} \notin \mathcal{bV}^{\mathcal{A}}(p) \).

So the inclusion is proper as asserted.

### IV. Conclusion

Convergence theory is used as a basic tool in, measure spaces, sequences of random variables, information theory etc. In this paper we have introduced and studied the notion of bounded variation difference multiple sequences \( \mathcal{bV}^{\mathcal{A}}(p) \) of fuzzy real numbers. We have verified some algebraic and topological properties of the space. Although we prove our results only for triple sequences, but all these results remain true for d-multiple sequences as well. The introduced notion can be applied for further investigations from different aspects.
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